SUFFICIENT CONDITIONS FOR THE OSCILLATION OF DELAY DIFFERENCE EQUATIONS

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ABSTRACT. The most important result of this paper is a new oscillation criterion for delay difference equations. This criterion constitutes a substantial improvement of the one by Ladas, Philos and Sficas [*J. Appl. Math. Simulation* **2** (1989), 101-111] and should be looked upon as the discrete analogue of a well-known oscillation criterion for delay differential equations.

1. INTRODUCTION

In the last two decades, the study of difference equations has been the focus of great attention by many researchers. Besides its mathematical interest, the theory of difference equations is also very interesting because of the fact that difference equations arise in various fields of applied sciences more frequently than ever. In particular, the study of the oscillation of solutions of difference equations has attracted a lot of activity.

It is the main purpose of this paper to establish a new oscillation criterion for linear delay difference equations with variable coefficients. This criterion substantially improves the one by Ladas, Philos and Sficas [16] and should be looked upon as the discrete analogue of a well-known integral oscillation result (see Ladas [12], and Koplatadze and Chanturija [11]) for first order linear delay differential equations with variable coefficients.

Consider the delay difference equation

$$x_{n+1} - x_n + p_n x_{n-k} = 0, \tag{E}$$

where $(p_n)_{n\geq 0}$ *is a sequence of nonnegative real numbers and* k *is a positive integer.*

By a *solution* of (E), we mean a sequence $(x_n)_{n\geq -k}$ of real numbers which satisfies (E) for all $n \geq 0$.

A solution $(x_n)_{n\geq -k}$ of (E) is said to be *oscillatory* if the terms x_n of the sequence are neither eventually positive nor eventually negative, and otherwise the solution is called *nonoscillatory.*

In 1989, Erbe and Zhang [9] established that *all solutions of* (E) *are oscillatory if*

$$\liminf_{n\to\infty} p_n > \frac{k^k}{(k+1)^{k+1}} \tag{H$_0$}$$

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or

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n} p_i > 1. \tag{G}$$

In the same year 1989, Ladas, Philos and Sficas [16] proved that *a sufficient condition for all solutions of* (E) *to be oscillatory is that*

$$\liminf_{n\to\infty} \left(\frac{1}{k} \sum_{i=n-k}^{n-1} p_i \right) > \frac{k^k}{(k+1)^{k+1}} . \tag{H}$$

Observe that condition (H) improves (H_0).

In particular, let us consider the special case where (E) is autonomous, i.e. the case of the delay difference equation

$$x_{n+1} - x_n + p x_{n-k} = 0, \tag{E_0}$$

where *p is a positive real number and k is a positive integer.* In this special case, both conditions (H_0) and (H) reduce to

$$p > \frac{k^k}{(k+1)^{k+1}} , \tag{h}$$

which *is a necessary and sufficient condition for the oscillation of all solutions of* (E_0). See Ladas [13]; see, also, Erbe and Zhang [9], Ladas [14], and Ladas, Philos and Sficas [16].

Consider now the delay differential equation

$$x'(t) + p(t)x(t-\tau) = 0, \tag{D}$$

where p is a nonnegative continuous real-valued function on the interval $[0, \infty)$ and τ is a positive constant.

It is well-known (see Myskis [21]; see, also, [10] or [17]) that the condition

$$\tau \liminf_{t\to\infty} p(t) > \frac{1}{e} \tag{A_0}$$

implies the oscillation of all solutions of (D). Another condition, which ensures that all solutions of (D) are oscillatory, is

$$\limsup_{t\to\infty} \int_{t-\tau}^{t} p(s)ds > 1. \tag{B}$$

(See Ladas, Lakshmikantham and Papadakis [15]; see, also, [10] or [17].)

It is noteworthy that conditions (H_0) and (G) are the discrete versions of (A_0) and (B) respectively.

Furthermore, a sharp sufficient condition for all solutions of (D) to be oscillatory is that

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} p(s)ds > \frac{1}{e} ; \tag{A}$$

see Ladas [12], and Koplatadze and Chanturija [11]. It is obvious that condition (A) is implied by (A_0).

As it has been pointed out in [16], condition (H) is a discrete analogue of (A). However, in order to prove that (H) is a sufficient condition for all solutions of (E) to oscillate, the authors in [16] use the well-known inequality of the arithmetic and geometric means. Note that, in the continuous case, in order to prove that (A) is a sufficient condition for the oscillation of all solutions of (D), no such (analogue) inequality is used. Due to this fact, it seems that the procedure used to establish (H) in [16] may not be the exact discrete analogue of the one used to establish condition (A).

Following the discrete version of the procedure used in the continuous case and without using the arithmetic mean - geometric mean inequality, we establish here a new oscillation criterion for the difference equation (E), which may be considered as the discrete analogue of the oscillation condition (A) for the differential equation (D). Note that our oscillation criterion essentially improves the oscillation result by Ladas, Philos and Sficas in [16].

Since 1989 and motivated by the results in [9] and [16], a number of related papers has been published. We choose to refer to the following papers (presented in chronological order): Philos [22, 23], Lalli and Zhang [18], Yu, Zhang and Qian [32], Chen and Yu [2], Cheng and Zhang [5], Domshlak [7], Yu, Zhang and Wang [33], Cheng, Xi and Zhang [4], Stavroulakis [25], Zhang, Liu and Cheng [35], Wong and Agarwal [34], (see also [1]), Tang [26], Cheng, Liu and Zhang [3], Domshlak [8], Tang and Yu [27, 28, 29], Cheng and Zhang [6], Yu and Tang [31], Luo and Shen [19,2 0], Shen and Stavroulakis [24], Tang and Zhang [30], and the references cited therein.

The paper is organized as follows. Section 2 is devoted to the statement of our results and to some extensive comments about them. The proofs of our results are given in Section 3. Section 4 contains the application of the results to the special case of periodic delay difference equations. Two examples, in which our oscillation criterion can be applied while condition (H) fails to hold, are presented in Section 5. In the last section (Section 6), the analogues results for advanced difference equations are formulated (without their proofs).

2. STATEMENT OF THE RESULTS AND COMMENTS

Our first result is the following proposition.

Proposition. (i) *A necessary condition for* (E) *to have at least one nonoscillatory solution is that*

$$(\mathrm{P}_0) \qquad p_n < 1 \quad \textit{for all large } n.$$

(ii) *Let* (P_0) *be satisfied and assume that*

$$(\mathrm{C}_1) \qquad \limsup_{n\to\infty} \left[\prod_{i=n-k}^{n-1} (1-p_i) \right]^{1/k} < 1 - \frac{k^k}{(k+1)^{k+1}} \,.$$

Then a necessary condition for (E) *to have at least one nonoscillatory solution is that*

$$(\mathrm{P}) \qquad \limsup_{n\to\infty} p_n < 1.$$

Note: Clearly, (P) implies (P_0).

The above proposition can equivalently be stated as follows:

(i)′ *All solutions of* (E) *are oscillatory if* (P_0) *fails.*

(ii)′ *Let* (P_0) *be satisfied and assume that condition* (C_1) *holds. All solutions of* (E) *are oscillatory if* (P) *fails, i.e. if* $\limsup_{n\to\infty} p_n = 1$.

Based on this result, while looking for sufficient conditions for the oscillation of all solutions of (E), we can see that it suffices to consider the case where (P_0) holds only. Furthermore, provided that (P_0) is satisfied and, in addition, condition (C_1) holds, we may confine our discussion only to the case where (P) is fulfilled.

Note that (P) is equivalent to the condition: *There exist numbers* $\gamma \in (0,1)$ *such that* $p_n < \gamma$ *for all large* n.

Next, the following basic lemma is stated.

Lemma 1. *Let* (P) *be satisfied and assume that condition* (C_1) *holds and that:*
(C_2) *There exists a constant* $\mu \in (0,1)$ *so that, for any number* $\gamma \in (0,1)$ *such that* $p_n < \gamma$ *for all large* n, *it holds*

$$\prod_{i=n-k}^{n-1}\left(1-\frac{1}{\gamma}p_i\right) \le \mu\gamma \quad \textit{for all large } n.$$

Then, for every nonoscillatory solution $(x_n)_{n\ge -k}$ *of* (E), *we have*

$$\lim_{n\to\infty}\frac{x_{n-k}}{x_n}=\infty.$$

The proof of the following result is contained in the proof of Theorem 1 in the paper by Ladas, Philos and Sficas [16] (see, also, the proof of Theorem 7.5.1 in the book by Györi and Ladas [10]).

Assume that there exists a positive constant M *such that*

$$\sum_{i=n-k}^{n} p_i \ge M \quad \textit{for all large } n$$

and let $(x_n)_{n\ge -k}$ *be a nonoscillatory solution of* (E).

Then, for each sufficiently large n, *there exists an integer* n^* *with* $n-k \le n^* \le n$ *so that*

$$\frac{x_{n^*-k}}{x_{n^*}} \le \frac{4}{M^2}.$$

In some papers, a false statement has been used instead of the above (correct) result. This fact has leaded to certain erroneous proofs as well as to some incorrect results. For a detailed discussion, the reader is referred to Domshlak [8], and Cheng and Zhang [6].

The following lemma is an immediate consequence of the above result.

Lemma 2. *Assume that*

$$(\mathrm{C}_3) \qquad \liminf_{n\to\infty}\sum_{i=n-k}^{n} p_i > 0.$$

Then, for every nonoscillatory solution $(x_n)_{n \geq -k}$ *of* (E), *there exists a sequence of nonnegative integers* $(n_\nu)_{\nu \geq 0}$ *with* $\lim_{\nu \to \infty} n_\nu = \infty$ *so that*

$$\limsup_{\nu \to \infty} \frac{x_{n_\nu - k}}{x_{n_\nu}} < \infty.$$

In [8], Domshlak presented the discrete analogue of the well-known Koplatadze-Chanturija lemma (see [11]) for delay differential equations. More precisely, it has been proved in [8] that, if M is a positive constant such that

$$\sum_{i=n-k}^{n-1} p_i \geq M \quad \text{for all large } n,$$

then every nonoscillatory solution $(x_n)_{n \geq -k}$ of (E) satisfies

$$\frac{x_{n-k}}{x_n} < \frac{4}{M^2} \quad \text{for all large } n.$$

A similar result with the constant M^{-k} in place of $4/M^2$ has been given by Cheng and Zhang [6]. So, it follows that, *under the condition*

$$(\mathrm{C}_3') \qquad \liminf_{n \to \infty} \sum_{i=n-k}^{n-1} p_i > 0,$$

every nonoscillatory solution $(x_n)_{n \geq -k}$ *of* (E) *satisfies*

$$\limsup_{n \to \infty} \frac{x_{n-k}}{x_n} < \infty.$$

Note that (C_3') is stronger than condition (C_3).

A combination of Lemmas 1 and 2 leads to the following oscillation criterion for the solutions of (E). This oscillation criterion is the most important result of our paper.

Theorem. *Let* (P) *be satisfied and assume that conditions* (C_1) *and* (C_2) *hold. Moreover, suppose that condition* (C_3) *holds.*

Then all solutions of (E) *are oscillatory.*

Consider the delay differential inequalities

$$(\mathrm{I})_{\leq} \qquad x_{n+1} - x_n + p_n x_{n-k} \leq 0$$

and

$$(\mathrm{I})_{\geq} \qquad x_{n+1} - x_n + p_n x_{n-k} \geq 0.$$

By a *solution* of $(\mathrm{I})_{\leq}$ [respectively, of $(\mathrm{I})_{\geq}$], we mean a sequence of real numbers $(x_n)_{n \geq -k}$ which satisfies $(\mathrm{I})_{\leq}$ [resp., $(\mathrm{I})_{\geq}$] for all $n \geq 0$.

Our theorem can be stated in a more general form as follows:

Let (P) *be satisfied and assume that conditions* (C_1) *and* (C_2) *hold. Moreover, suppose that condition* (C_3) *holds.*

Then $(\mathrm{I})_{\leq}$ *has no eventually positive solutions, and* $(\mathrm{I})_{\geq}$ *has no eventually negative solutions.*

As far as the existence of nonoscillatory solutions of (E) is concerned, we have the following result established by Ladas, Philos and Sficas [16]:

Assume that

$$\sum_{j=0}^{k-1} p_{n+j} > 0 \quad \textit{for } n \geq 0$$

and let there exist a number $\gamma \in (0,1)$ *so that*

$$p_n < \gamma \quad \textit{for every } n \geq 0$$

and

$$\prod_{i=n-k}^{n-1} \left(1 - \frac{1}{\gamma}\widetilde{p}_i\right) \geq \gamma \quad \textit{for all } n \geq 0,$$

where

$$\widetilde{p}_n = p_n \quad \textit{for } n \geq 0, \qquad \textit{and} \qquad \widetilde{p}_n = p_0 \quad \textit{for } -k \leq n < 0.$$

Then (E) *has a positive solution* $(x_n)_{n \geq -k}$ *such that*

$$\lim_{n \to \infty} x_n = 0.$$

In the special case of the difference equation (E_0), the assumptions of the above result are satisfied if

$$p \leq \frac{k^k}{(k+1)^{k+1}}$$

(see Ladas, Philos and Sficas [16]). Note that the last condition is also a necessary condition for the existence of a positive solution of (E).

Consider now the special case of the difference equation (E_0). In this case, both conditions (P_0) and (P) reduce to

$$p < 1.$$

Moreover, we immediately see that condition (C_3) holds by itself. We can also verify that, when $p < 1$, condition (C_1) is equivalent to (h). Furthermore, provided that $p < 1$, (h) implies condition (C_2). To show this fact, we first observe that

$$\max_{\lambda \in [0,1]} \left[\lambda(1-\lambda)^k\right] = \left[\lambda(1-\lambda)^k\right]_{\lambda = 1/(k+1)} = \frac{k^k}{(k+1)^{k+1}}$$

and so

$$(1-\lambda)^k \leq \frac{1}{\lambda} \cdot \frac{k^k}{(k+1)^{k+1}} \quad \text{for } \lambda \in (0,1). \tag{I}$$

This inequality will also be used below in this section as well as in the next section. Let us suppose that $p < 1$ and that (h) is satisfied. Set

$$\mu = \frac{1}{p} \cdot \frac{k^k}{(k+1)^{k+1}}\ .$$

(Clearly, μ is a constant with $\mu \in (0,1)$.) Let $\gamma \in (0,1)$ be an arbitrary number with $p < \gamma$. Then, by using the inequality (I), we find

$$\left(1 - \frac{1}{\gamma}p\right)^k \leq \frac{1}{\frac{1}{\gamma}p} \cdot \frac{k^k}{(k+1)^{k+1}} = \left[\frac{1}{p} \cdot \frac{k^k}{(k+1)^{k+1}}\right]\gamma = \mu\gamma.$$

Hence, (C_2) is always fulfilled.

From our proposition it follows that $p < 1$ is a necessary condition for (E_0) to have at least one nonoscillatory solution. Thus, all solutions of (E_0) are oscillatory if $p \geq 1$. Moreover, when $p < 1$, our theorem guarantees that (h) suffices for the oscillation of all solutions of (E_0). So, we have arrived at the well-known result that (h) is a sufficient condition for all solutions of (E_0) to be oscillatory. Note that (h) is also a necessary condition in order that all solutions of (E_0) to be oscillatory.

Next, we will show that, *provided that* (P) *holds, condition* (H) *implies conditions* (C_1) *and* (C_2). To this end, let us assume that (P) holds and that condition (H) is satisfied. By using the well-known inequality of the arithmetic and geometric means, we find for all large n

$$\left[\prod_{i=n-k}^{n-1}(1-p_i)\right]^{1/k} \leq \frac{1}{k}\sum_{i=n-k}^{n-1}(1-p_i) = 1-\frac{1}{k}\sum_{i=n-k}^{n-1}p_i$$

and consequently, by taking into account (H), we obtain

$$\limsup_{n\to\infty}\left[\prod_{i=n-k}^{n-1}(1-p_i)\right]^{1/k} \leq 1-\liminf_{n\to\infty}\left(\frac{1}{k}\sum_{i=n-k}^{n-1}p_i\right) < 1-\frac{k^k}{(k+1)^{k+1}},$$

which means condition (C_1) holds. Furthermore, because of (H), we can choose a constant $\mu \in (0,1)$ so that

$$\frac{1}{k}\sum_{i=n-k}^{n-1}p_i \geq \frac{1}{\mu}\cdot\frac{k^k}{(k+1)^{k+1}} \quad \text{for all large } n.$$

Consider an arbitrary number $\gamma \in (0,1)$ such that $p_n < \gamma$ for all large n, i.e. such that $1-\frac{1}{\gamma}p_n > 0$ for n sufficiently large. By using again the inequality of the arithmetic and geometric means, we obtain for all large n

$$\begin{aligned}\prod_{i=n-k}^{n-1}\left(1-\frac{1}{\gamma}p_i\right) &\leq \left[\frac{1}{k}\sum_{i=n-k}^{n-1}\left(1-\frac{1}{\gamma}p_i\right)\right]^k = \left[1-\frac{1}{\gamma}\left(\frac{1}{k}\sum_{i=n-k}^{n-1}p_i\right)\right]^k \\ &\leq \left[1-\frac{1}{\gamma}\left(\frac{1}{\mu}\cdot\frac{k^k}{(k+1)^{k+1}}\right)\right]^k = \left[1-\frac{1}{\mu\gamma}\cdot\frac{k^k}{(k+1)^{k+1}}\right]^k.\end{aligned}$$

Since $\gamma > p_n$ for all large n, we have eventually

$$\gamma > \frac{1}{k}\sum_{i=n-k}^{n-1}p_i \geq \frac{1}{\mu}\cdot\frac{k^k}{(k+1)^{k+1}}$$

and so

$$0 < \frac{1}{\mu\gamma}\cdot\frac{k^k}{(k+1)^{k+1}} < 1.$$

Thus, an application of (I) gives

$$\left[1-\frac{1}{\mu\gamma}\cdot\frac{k^k}{(k+1)^{k+1}}\right]^k \leq \frac{1}{\frac{1}{\mu\gamma}\cdot\frac{k^k}{(k+1)^{k+1}}}\cdot\frac{k^k}{(k+1)^{k+1}} = \mu\gamma.$$

Hence, it holds

$$\prod_{i=n-k}^{n-1}\left(1-\frac{1}{\gamma}p_i\right) \le \mu\gamma \quad \text{for all large } n$$

and consequently condition (C_2) is also satisfied.

Finally, since

$$\sum_{i=n-k}^{n} p_i \ge \sum_{i=n-k}^{n-1} p_i = k\left(\frac{1}{k}\sum_{i=n-k}^{n-1} p_i\right) \quad \text{for all large } n,$$

it follows immediately that *condition* (H) *implies condition* (C_3).

In Section 5, we will give two examples, in which (P) is satisfied and conditions (C_1), (C_2) and (C_3) hold, while condition (H) fails to hold.

3. PROOF OF THE PROPOSITION AND LEMMA 1

Let $(x_n)_{n\ge -k}$ be a nonoscillatory solution of (E). As the negative of a solution of (E) is also a solution of (E), we may (and do) assume that $(x_n)_{n\ge -k}$ is eventually positive. Then eventually

$$x_{n+1} - x_n = -p_n x_{n-k} \le 0$$

and so $(x_n)_{n\ge -k}$ is eventually decreasing. It follows from (E) that eventually

$$x_{n+1} - x_n = -p_n x_{n-k} \le -p_n x_n$$

and consequently we have

$$\frac{x_{n+1}}{x_n} \le 1 - p_n \text{ for all large } n.$$

In particular, this implies that $1 - p_n > 0$ for all large n, i.e. (P_0) holds true (and so Part (i) of our proposition has been proved).

Next, assume that condition (C_1) holds. Then we can choose a number γ_0 with $\frac{k^k}{(k+1)^{k+1}} < \gamma_0 < 1$ so that

$$\left[\prod_{i=n-k}^{n-1}(1-p_i)\right]^{1/k} \le 1 - \frac{1}{\gamma_0}\cdot\frac{k^k}{(k+1)^{k+1}} \quad \text{for all large } n.$$

Thus, we obtain for n sufficiently large

$$\frac{x_n}{x_{n-k}} = \prod_{i=n-k}^{n-1}\frac{x_{i+1}}{x_i} \le \prod_{i=n-k}^{n-1}(1-p_i) \le \left[1 - \frac{1}{\gamma_0}\cdot\frac{k^k}{(k+1)^{k+1}}\right]^k.$$

Since $0 < \frac{1}{\gamma_0}\cdot\frac{k^k}{(k+1)^{k+1}} < 1$, from (I) it follows that

$$\left[1 - \frac{1}{\gamma_0}\cdot\frac{k^k}{(k+1)^{k+1}}\right]^k \le \frac{1}{\frac{1}{\gamma_0}\cdot\frac{k^k}{(k+1)^{k+1}}}\cdot\frac{k^k}{(k+1)^{k+1}} = \gamma_0.$$

Hence, we get

$$\frac{x_n}{x_{n-k}} \le \gamma_0 \text{ for } n \text{ sufficiently large,}$$

i.e.

$$\frac{x_{n-k}}{x_n} \geq \frac{1}{\gamma_0} \text{ for all large } n.$$

Furthermore, (E) gives eventually

$$x_{n+1} - x_n = -p_n x_{n-k} \leq -\frac{1}{\gamma_0}\, p_n x_n$$

and so we have

$$\frac{x_{n+1}}{x_n} \leq 1 - \frac{1}{\gamma_0} p_n \text{ for all large } n.$$

In particular, this guarantees that $1 - \frac{1}{\gamma_0} p_n > 0$ for all large n, i.e. $p_n < \gamma_0$ for n sufficiently large. This ensures that $\limsup_{n\to\infty} p_n \leq \gamma_0 < 1$ and consequently (P) is satisfied (and so Part (ii) of the proposition has been showed).

Now, let us suppose that condition (C_2) is also satisfied. By this condition (for $\gamma = \gamma_0$), we find that eventually

$$\frac{x_n}{x_{n-k}} = \prod_{i=n-k}^{n-1} \frac{x_{i+1}}{x_i} \leq \prod_{i=n-k}^{n-1} \left(1 - \frac{1}{\gamma_0} p_i\right) \leq \mu\gamma_0.$$

Thus, we have

$$\frac{x_{n-k}}{x_n} \geq \frac{1}{\mu\gamma_0} \text{ for all large } n.$$

Hence, it follows from (E) that eventually

$$x_{n+1} - x_n = -p_n x_{n-k} \leq -\frac{1}{\mu\gamma_0} x_n$$

and consequently

$$\frac{x_{n+1}}{x_n} \leq 1 - \frac{1}{\mu\gamma_0} p_n \text{ for all large } n.$$

So, we always have $1 - \frac{1}{\mu\gamma_0} p_n > 0$ eventually, i.e. it holds $p_n < \mu\gamma_0$ for all large n. Thus, because of condition (C_2) (for $\gamma = \mu\gamma_0$), we obtain, for n sufficiently large,

$$\frac{x_n}{x_{n-k}} = \prod_{i=n-k}^{n-1} \frac{x_{i+1}}{x_i} \leq \prod_{i=n-k}^{n-1} \left(1 - \frac{1}{\mu\gamma_0} p_i\right) \leq \mu(\mu\gamma_0) = \mu^2\gamma_0.$$

Therefore,

$$\frac{x_{n-k}}{x_n} \geq \frac{1}{\mu^2\gamma_0} \text{ for all large } n.$$

Finally, by induction, it follows that

$$\frac{x_{n-k}}{x_n} \geq \frac{1}{\mu^m\gamma_0} \text{ for all large } n \qquad (m = 0, 1, 2, \ldots).$$

Since $\mu \in (0, 1)$, we have $\lim_{m\to\infty} \mu^m = 0$. Hence, we conclude that

$$\lim_{n\to\infty} \frac{x_{n-k}}{x_n} = \infty$$

(and so the proof of Lemma 1 is complete).

4. THE SPECIAL CASE OF PERIODIC DELAY DIFFERENCE EQUATIONS

Let us concentrate our interest on the special case where the sequence $(p_n)_{n\geq 0}$ is k-periodic.

Both conditions ($\mathrm{P_0}$) and (P) reduce to

$$(\widetilde{\mathrm{P}}) \qquad p_r < 1 \quad (r = 0, 1, ..., k-1).$$

By our proposition, we have the following result:

Assume that the coefficient sequence $(p_n)_{n\geq 0}$ is k-periodic. Then $(\widetilde{\mathrm{P}})$ is a necessary condition for (E) *to have at least one nonoscillatory solution.*

Moreover, provided that $(\widetilde{\mathrm{P}})$ is satisfied, conditions ($\mathrm{C_1}$) and ($\mathrm{C_2}$) become respectively

$$(\widetilde{\mathrm{C}}_1) \qquad \left[\prod_{r=0}^{k-1}(1-p_r)\right]^{1/k} < 1 - \frac{k^k}{(k+1)^{k+1}}$$

and

$(\widetilde{\mathrm{C}}_2)$ *There exists a constant $\mu \in (0,1)$ so that, for any number $\gamma \in (0,1)$ such that $p_r < \gamma$ $(r = 0, 1, ..., k-1)$, it holds*

$$\prod_{r=0}^{k-1}\left(1 - \frac{1}{\gamma}p_r\right) \leq \mu\gamma.$$

Furthermore, we observe that

$$\sum_{i=n-k}^{n} p_i \geq \sum_{i=n-k}^{n-1} p_i = \sum_{r=0}^{k-1} p_r \quad \text{for all large } n$$

and so, provided that $(p_n)_{n\geq 0}$ is not identically zero, condition ($\mathrm{C_3}$) holds by itself. Hence, our theorem leads to the following result:

Assume that the coefficient sequence $(p_n)_{n\geq 0}$ is k-periodic, and let $(\widetilde{\mathrm{P}})$ be satisfied. Then conditions $(\widetilde{\mathrm{C}}_1)$ and $(\widetilde{\mathrm{C}}_2)$ are sufficient for all solutions of (E) *to be oscillatory.*

5. EXAMPLES

In this section, we will give two examples, in which *all assumptions of our theorem (i.e. conditions* (P), ($\mathrm{C_1}$), ($\mathrm{C_2}$) *and* ($\mathrm{C_3}$)*) are satisfied while condition* (H) *fails.*

Example 1. *Consider the delay difference equation* (E) *with $k > 1$ and assume that the coefficient sequence $(p_n)_{n\geq 0}$ is k-periodic with*

$$p_0 = \delta \quad \text{and} \quad p_1 = p_2 = ... = p_{k-1} = 0,$$

where δ is a constant such that

$$1 - \left[1 - \frac{k^k}{(k+1)^{k+1}}\right]^k < \delta \leq k \cdot \frac{k^k}{(k+1)^{k+1}}\,.$$

Note that it holds

$$0 < 1 - \left[1 - \frac{k^k}{(k+1)^{k+1}}\right]^k < k \cdot \frac{k^k}{(k+1)^{k+1}} < 1$$

(and so $0 < \delta < 1$). We immediately observe that (P) becomes

$$p_r < 1 \quad (r = 0, 1, ..., k-1)$$

and consequently *condition* (P) *is always satisfied.* Moreover, (C_1) reduces to

$$\left[\prod_{r=0}^{k-1}(1 - p_r)\right]^{1/k} < 1 - \frac{k^k}{(k+1)^{k+1}}, \quad \text{i.e.} \quad (1-\delta)^{1/k} < 1 - \frac{k^k}{(k+1)^{k+1}}$$

or

$$\delta > 1 - \left[1 - \frac{k^k}{(k+1)^{k+1}}\right]^k$$

and so *condition* (C_1) *is also satisfied.* Now, we observe that (C_2) becomes: There exists a constant $\mu \in (0,1)$ so that, for any number $\gamma \in (\delta, 1)$, it holds

$$\prod_{r=0}^{k-1}\left(1 - \frac{1}{\gamma}p_r\right) \equiv 1 - \frac{\delta}{\gamma} \leq \mu\gamma, \quad \text{i.e.} \quad \frac{1}{\gamma}\left(1 - \frac{\delta}{\gamma}\right) \leq \mu.$$

Define

$$f(\gamma) = \frac{1}{\gamma}\left(1 - \frac{\delta}{\gamma}\right) \quad \text{for } \gamma \in (\delta, 1).$$

Then

$$f'(\gamma) = -\frac{1}{\gamma^3}(\gamma - 2\delta) \quad \text{for } \gamma \in (\delta, 1).$$

But, it is not difficult to see that

$$k \cdot \frac{k^k}{(k+1)^{k+1}} < \frac{1}{2}.$$

This guarantees that $\delta < \frac{1}{2}$ and consequently we have

$$\delta < 2\delta < 1.$$

So,

$$f'(\gamma) > 0 \text{ for } \gamma \in (\delta, 2\delta), \quad \text{and} \quad f'(\gamma) < 0 \text{ for } \gamma \in (2\delta, 1)$$

and hence f is strictly increasing on $(\delta, 2\delta)$, and f is strictly decreasing on $(2\delta, 1)$. Thus, (C_2) holds if and only if

$$0 < \mu \equiv f(2\delta) < 1,$$

i.e. if and only if

$$\delta > \frac{1}{4}.$$

It is a matter of elementary calculus to show that

$$\left[1 - \frac{k^k}{(k+1)^{k+1}}\right]^k < \frac{3}{4}.$$

(Note that $k > 1$.) This yields

$$\delta > 1 - \left[1 - \frac{k^k}{(k+1)^{k+1}}\right]^k > 1 - \frac{3}{4} = \frac{1}{4}.$$

Hence, *condition* (C_2) *is satisfied.* Next, we see that *condition* (C_3) *holds by itself.* On the other hand, (H) becomes

$$\frac{1}{k}\sum_{r=0}^{k-1} p_r > \frac{k^k}{(k+1)^{k+1}}, \quad \text{i.e.} \quad \frac{1}{k}\delta > \frac{k^k}{(k+1)^{k+1}}$$

or

$$\delta > k \cdot \frac{k^k}{(k+1)^{k+1}}$$

and consequently *condition* (H) *fails to hold.*

Example 2. *Consider the delay difference equation* (E) *and assume that* $k = 2$ *and that the coefficient sequence* $(p_n)_{n \geq 0}$ *is 2-periodic with*

$$p_0 = \epsilon \quad \textit{and} \quad p_1 = \frac{8}{27} - \epsilon,$$

where ϵ *is a number such that*

$$\frac{4}{27} < \epsilon < \frac{8}{27}.$$

We have

$$0 < p_1 < \frac{4}{27} < p_0 < \frac{8}{27} < 1.$$

We see that (P) becomes

$$p_0 < 1 \quad \text{and} \quad p_1 < 1$$

and so *condition* (P) *holds true.* We also observe that (C_1) reduces to

$$[(1-p_0)(1-p_1)]^{1/2} < 1 - \frac{4}{27} = \frac{23}{27},$$

i.e.

$$(1-\epsilon)\left[1 - \left(\frac{8}{27} - \epsilon\right)\right] < \left(\frac{23}{27}\right)^2,$$

which can equivalently be written

$$\left(\epsilon - \frac{4}{27}\right)^2 > 0.$$

This means that *condition* (C_1) *is always satisfied.* Next, we will show that condition (C_2) is also satisfied. To this end, we see that (C_2) becomes: There exists a constant $\mu \in (0,1)$ so that, for any number $\gamma \in (\epsilon, 1)$, it holds

$$\left(1 - \frac{p_0}{\gamma}\right)\left(1 - \frac{p_1}{\gamma}\right) \leq \mu\gamma$$

i.e.

$$\frac{1}{\gamma}\left(1 - \frac{\epsilon}{\gamma}\right)\left(1 - \frac{\frac{8}{27} - \epsilon}{\gamma}\right) \leq \mu.$$

Let us define

$$F(\gamma) = \frac{1}{\gamma}\left(1 - \frac{\epsilon}{\gamma}\right)\left(1 - \frac{\frac{8}{27} - \epsilon}{\gamma}\right) \quad \text{for } \gamma \in (\epsilon, 1).$$

Then we find

$$F'(\gamma) = -\frac{1}{\gamma^4}\left[\gamma^2 - \frac{16}{27}\gamma + 3\epsilon\left(\frac{8}{27} - \epsilon\right)\right] \quad \text{for } \gamma \in (\epsilon, 1).$$

Consider the quadratic equation

$$\gamma^2 - \frac{16}{27}\gamma + 3\epsilon\left(\frac{8}{27} - \epsilon\right) = 0$$

in the complex plane. It is easy to see that this equation has exactly two distinct real roots given by

$$\gamma^* = \frac{8}{27} - \sqrt{\left(\frac{8}{27}\right)^2 - 3\epsilon\left(\frac{8}{27} - \epsilon\right)} \quad \text{and} \quad \gamma_0 = \frac{8}{27} + \sqrt{\left(\frac{8}{27}\right)^2 - 3\epsilon\left(\frac{8}{27} - \epsilon\right)}.$$

[Note that $\left(\frac{8}{27}\right)^2 - 3\epsilon\left(\frac{8}{27} - \epsilon\right) > 0$.] It follows immediately that $\gamma^* > 0$ and $\gamma_0 < \frac{16}{27}$. Furthermore, by using the inequality $\epsilon\left(\frac{8}{27} - \epsilon\right) < \left(\frac{4}{27}\right)^2$, we can see that $\gamma^* < \frac{4}{27}$ and $\gamma_0 > \frac{12}{27}$. Thus, we have $0 < \gamma^* < \frac{4}{27}$ and $\frac{12}{27} < \gamma_0 < \frac{16}{27}$ and so it holds

$$0 < \gamma^* < \epsilon < \gamma_0 < 1.$$

Hence, we conclude that

$$\gamma^2 - \frac{16}{27}\gamma + 3\epsilon\left(\frac{8}{27} - \epsilon\right) < 0 \quad \text{for } \gamma \in (\epsilon, \gamma_0)$$

and

$$\gamma^2 - \frac{16}{27}\gamma + 3\epsilon\left(\frac{8}{27} - \epsilon\right) > 0 \quad \text{for } \gamma \in (\gamma_0, 1)$$

and consequenlty

$$F'(\gamma) > 0 \;\text{ for } \gamma \in (\epsilon, \gamma_0), \qquad \text{and} \qquad F'(\gamma) < 0 \;\text{ for } \gamma \in (\gamma_0, 1).$$

This means that F is strictly increasing on (ϵ, γ_0), and F is strictly decreasing on $(\gamma_0, 1)$. So, condition (C_2) holds if and only if

$$0 < F(\gamma_0) < 1.$$

Clearly, $F(\gamma_0) > 0$. Furthermore, we obtain

$$\begin{aligned}
F(\gamma_0) &\equiv \frac{1}{\gamma_0}\left(1 - \frac{\epsilon}{\gamma_0}\right)\left(1 - \frac{\frac{8}{27} - \epsilon}{\gamma_0}\right) = \frac{1}{\gamma_0^3}(\gamma_0 - \epsilon)\left[\gamma_0 - \left(\frac{8}{27} - \epsilon\right)\right] \\
&= \frac{1}{\gamma_0^3}\left[\gamma_0^2 - \frac{8}{27}\gamma_0 + \epsilon\left(\frac{8}{27} - \epsilon\right)\right] = \frac{1}{3\gamma_0^3}\left[3\left(\gamma_0^2 - \frac{8}{27}\gamma_0\right) + 3\epsilon\left(\frac{8}{27} - \epsilon\right)\right] \\
&= \frac{1}{3\gamma_0^3}\left[3\left(\gamma_0^2 - \frac{8}{27}\gamma_0\right) - \left(\gamma_0^2 - \frac{16}{27}\gamma_0\right)\right] = \frac{1}{3\gamma_0^3}\left(2\gamma_0^2 - \frac{8}{27}\gamma_0\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2\gamma_0 - \frac{8}{27}}{3\gamma_0^2} = \frac{2\gamma_0 - \frac{8}{27}}{3\left[\frac{16}{27}\gamma_0 - 3\epsilon\left(\frac{8}{27} - \epsilon\right)\right]} \\
&= \frac{2\left[\frac{8}{27} + \sqrt{\left(\frac{8}{27}\right)^2 - 3\epsilon\left(\frac{8}{27} - \epsilon\right)}\right] - \frac{8}{27}}{3\left\{\frac{16}{27}\left[\frac{8}{27} + \sqrt{\left(\frac{8}{27}\right)^2 - 3\epsilon\left(\frac{8}{27} - \epsilon\right)}\right] - 3\epsilon\left(\frac{8}{27} - \epsilon\right)\right\}} \\
&= \frac{\frac{8}{27} + 2\sqrt{\left(\frac{8}{27}\right)^2 - 3\epsilon\left(\frac{8}{27} - \epsilon\right)}}{\frac{16}{9}\cdot\frac{8}{27} + \frac{16}{9}\sqrt{\left(\frac{8}{27}\right)^2 - 3\epsilon\left(\frac{8}{27} - \epsilon\right)} - 9\epsilon\left(\frac{8}{27} - \epsilon\right)}.
\end{aligned}
$$

By this expression of $F(\gamma_0)$, it is a matter of elementary calculations to see that $F(\gamma_0) < 1$ if and only if

$$\left[\epsilon\left(\frac{8}{27} - \epsilon\right)\right]^2 - \frac{4}{81}\left[\epsilon\left(\frac{8}{27} - \epsilon\right)\right] + \frac{320}{81^3} > 0,$$

i.e. if and only if

$$\left[\epsilon\left(\frac{8}{27} - \epsilon\right) - \frac{16}{729}\right]\left[\epsilon\left(\frac{8}{27} - \epsilon\right) - \frac{20}{729}\right] > 0.$$

But, this holds true, since from the inequality $\epsilon\left(\frac{8}{27} - \epsilon\right) < \left(\frac{4}{27}\right)^2$ it follows that

$$\epsilon\left(\frac{8}{27} - \epsilon\right) < \frac{16}{729} < \frac{20}{729}.$$

We have thus proved that *condition* (C_2) *is satisfied.* Now, it is clear that *condition* (C_3) *holds by itself.* Finally, we observe that (H) reduces to

$$\frac{1}{2}(p_0 + p_1) > \frac{4}{27}, \quad \text{i.e.} \quad \frac{1}{2}\left[\epsilon + \left(\frac{8}{27} - \epsilon\right)\right] > \frac{4}{27}$$

and consequently *condition* (H) *fails to hold.*

6. ADVANCED DIFFERENCE EQUATIONS

Let us consider the advanced difference equation

$$\text{(E)}^* \qquad x_{n+1} - x_n - p_n x_{n+k} = 0,$$

where *it is supposed that*

$$k > 1.$$

By a *solution* of (E)*, we mean a sequence of real numbers $(x_n)_{n\geq 0}$ which satisfies (E)* for all $n \geq 0$.

By using similar arguments with the ones used in proving the previous results for the delay difference equation (E), we can establish the following results for the advanced difference equation (E)*. We will state these results without proofs.

Proposition A. (I) *A necessary condition for* (E)* *to have at least one nonoscillatory solution is* (P_0).

(II) *Let* (P_0) *be satisfied and assume that*

$$(\mathrm{C}_1)^* \qquad \limsup_{n\to\infty}\left[\prod_{i=n+1}^{n+k-1}(1-p_i)\right]^{1/(k-1)} < 1-\frac{(k-1)^{k-1}}{k^k}.$$

Then a necessary condition for $(\mathrm{E})^*$ *to have at least one nonoscillatory solution is* (P).

Lemma 1-A. *Let* (P) *be satisfied and assume that condition* $(\mathrm{C}_1)^*$ *holds and that*:

$(\mathrm{C}_2)^*$ *There exists a constant* $\mu\in(0,1)$ *so that, for any number* $\gamma\in(0,1)$ *such that* $p_n<\gamma$ *for all large* n, *it holds*

$$\prod_{i=n+1}^{n+k-1}\left(1-\frac{1}{\gamma}p_i\right)\le\mu\gamma \quad \textit{for all large } n.$$

Then, for every nonoscillatory solution $(x_n)_{n\ge 0}$ *of* $(\mathrm{E})^*$, *we have*

$$\lim_{n\to\infty}\frac{x_{n+k}}{x_{n+1}}=\infty.$$

Lemma 2-A. *Assume that*

$$(\mathrm{C}_3)^* \qquad \liminf_{n\to\infty}\sum_{i=n}^{n+k-1}p_i>0.$$

Then, for every nonoscillatory solution $(x_n)_{n\ge 0}$ *of* $(\mathrm{E})^*$, *there exists a sequence of nonnegative integers* $(n_\nu)_{\nu\ge 0}$ *with* $\lim_{\nu\to\infty}n_\nu=\infty$ *so that*

$$\limsup_{\nu\to\infty}\frac{x_{n_\nu+k}}{x_{n_\nu+1}}<\infty.$$

Theorem A. *Let* (P) *be satisfied and assume that conditions* $(\mathrm{C}_1)^*$ *and* $(\mathrm{C}_2)^*$ *hold. Moreover, suppose that condition* $(\mathrm{C}_3)^*$ *holds.*

Then all solutions of $(\mathrm{E})^*$ *are oscillatory.*

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